EVERY BIHARMONIC HYPERSURFACE IN EUCLIDEAN SPACE IS MINIMAL

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Abstract. We prove that every biharmonic hypersurface in Euclidean space is minimal. This follows from our main theorem: every biharmonic submanifold in Euclidean space with parallel normalized mean curvature is minimal.

1. Introduction

Chen conjectured that every biharmonic submanifold in Euclidean space is minimal [Chen, 1991]. Hasanis and Vlachos [1995] proved the hypersurface case in $\mathbb{E}^4$, and Fu, Hong and Zhan [2021; 2023] proved it in $\mathbb{E}^5$ and $\mathbb{E}^6$. In arbitrary dimensions, the hypersurface case is known under additional geometric assumptions, such as simplicity of the principal curvatures and irreducibility of the principal curvature frame [Koiso and Urakawa, 2018], or holonomicity [Bibi et al., 2025]. We prove the general hypersurface case of Chen's 1991 conjecture without additional assumptions:

Corollary. *Every biharmonic hypersurface in Euclidean space is minimal.*

The corollary follows from our main theorem on submanifolds with parallel normalized mean curvature (PNMC).[1] Yeğin Şen and Turgay [2018] proved that every biharmonic surface in $\mathbb{E}^4$ with PNMC is minimal. Chen [2019] extended the surface result to arbitrary ambient dimension and conjectured the nonexistence of biharmonic submanifolds with PNMC in Euclidean spaces. Şen and Turgay [2023] subsequently established the conjecture for submanifolds in $\mathbb{E}^5$ with PNMC. Our main theorem proves Chen's 2019 conjecture in every dimension and codimension:

Theorem. *Every biharmonic submanifold in Euclidean space with PNMC is minimal.*

The proof has a geometric and an analytic part. Assuming that the submanifold is not minimal, we use a parallel principal frame along a suitable geodesic to reduce the geometric equations to a finite system of ordinary differential equations. A monotonicity argument shows that all the unknown functions remain bounded, allowing the solution to be continued for all positive time. On the other hand, the scalar biharmonic equation forces one of these functions to grow without bound. This contradiction shows that the original submanifold must be minimal.

2. Preliminaries

Let $x : M^m \to \mathbb{E}^{m+r}$ be a smooth immersion of an m-dimensional manifold M into Euclidean $(m+r)$-space, where $m, r \geq 1$. The Euclidean inner product $\langle \cdot, \cdot \rangle$ induces a Riemannian metric g on M,

$$g_p(u, v) = \langle dx_p(u), dx_p(v) \rangle, \qquad u, v \in T_pM,$$

where dx_p is the differential of x at p.

[1] We impose the PNMC condition only where the mean-curvature vector is nonzero; see Section 2.

Denote by D the Euclidean connection, by ∇ the Levi-Civita connection of g, and by $\nabla^{\perp}$ the induced normal connection. For tangent vector fields X, Y and a normal vector field η, the Gauss and Weingarten formulas are

$$D_X dx(Y) = dx(\nabla_X Y) + B(X,Y),$$
$$D_X \eta = -dx(A_\eta X) + \nabla^{\perp}_X \eta.$$

Here B is the second fundamental form and A_η is the shape operator in the direction η. They satisfy

$$g(A_\eta X, Y) = \langle B(X,Y), \eta \rangle.$$

The symmetry of B implies that each A_η is self-adjoint with respect to g. Its squared norm is

$$|A_\eta|^2 = \sum_{i=1}^{m} g(A_\eta e_i, A_\eta e_i) = \operatorname{tr}(A_\eta^2),$$

where $e_1, \ldots, e_m$ is a local orthonormal tangent frame.

The mean-curvature vector field is

$$H = \frac{1}{m} \operatorname{tr}_g B = \frac{1}{m} \sum_{i=1}^{m} B(e_i, e_i).$$

The immersion is called *minimal* if H vanishes identically. The definition of H also gives

$$\operatorname{tr} A_\eta = m\langle H, \eta \rangle.$$

We take the Laplacian of (M, g) to be $\Delta = \operatorname{div} \operatorname{grad}$, and extend it componentwise to $\mathbb{E}^{m+r}$-valued functions. For a vector field V along the immersion, this gives

$$\Delta V = \sum_{i=1}^{m} \left(D_{e_i} D_{e_i} V - D_{\nabla_{e_i} e_i} V \right).$$

The Gauss formula gives $\Delta x = mH$, so the immersion is harmonic ($\Delta x = 0$) if and only if it is minimal. It is called *biharmonic* if

$$\Delta^2 x = 0, \qquad \text{or, equivalently,} \qquad \Delta H = 0.$$

On the open set $U = \{p \in M : H(p) \neq 0\}$, write $h = |H|$ and $\xi = H/h$. The field ξ is the normalized mean-curvature vector. We say that the immersion has *parallel normalized mean curvature* (PNMC) if ξ is parallel in the normal bundle:

$$\nabla^{\perp}_X \xi = 0 \qquad \text{for every tangent vector field } X \text{ on } U.$$

This condition is imposed only on U, so our convention includes minimal immersions. Chen [2019] assumes that H is nowhere zero in the definition of PNMC; with that convention, our theorem is a nonexistence result. For hypersurfaces, the PNMC condition is automatic, since every local unit normal is parallel in the normal bundle.

For a PNMC immersion, $\nabla^{\perp}_X H = X(h)\xi$, and the Weingarten formula gives

$$D_X H = X(h)\xi - h\, dx(A_\xi X).$$

Differentiating and taking the trace gives

$$\Delta H = (\Delta h)\xi - h \sum_{i=1}^{m} B(e_i, A_\xi e_i) - dx(2A_\xi \operatorname{grad} h + h \operatorname{div} A_\xi),$$

where

$$\operatorname{div} A_\xi := \sum_{i=1}^{m} (\nabla_{e_i} A_\xi) e_i.$$

To express this divergence in terms of h, we use the Codazzi equation. The covariant derivative of the second fundamental form is

$$(\nabla^\perp_X B)(Y,Z) = \nabla^\perp_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),$$

and the Codazzi equation for an immersion into Euclidean space is

$$(\nabla^\perp_X B)(Y,Z) = (\nabla^\perp_Y B)(X,Z).$$

Since ξ is parallel, this implies

$$(\nabla_X A_\xi)Y = (\nabla_Y A_\xi)X.$$

Thus A_ξ is a Codazzi tensor, with $\operatorname{tr} A_\xi = mh$. Tracing the Codazzi identity gives

$$\operatorname{div} A_\xi = \operatorname{grad}(\operatorname{tr} A_\xi) = m \operatorname{grad} h.$$

For a biharmonic immersion with PNMC, the tangential part of $\Delta H = 0$ and its inner product with ξ therefore give (compare Chen [2019, Theorem 2])

$$A_\xi \operatorname{grad} h = -\frac{m}{2} h \operatorname{grad} h, \tag{2.1}$$

$$\Delta h = h|A_\xi|^2. \tag{2.2}$$

The second equation uses

$$\sum_i \langle B(e_i, A_\xi e_i), \xi\rangle = |A_\xi|^2.$$

These equations hold on U. Wherever $\operatorname{grad} h \neq 0$, the first equation identifies $\operatorname{grad} h$ as a principal direction of A_ξ, with principal curvature $-mh/2$.

To determine how the other normal shape operators act on the eigenspaces of A_ξ, we use the Ricci equation. For the curvature of the normal connection, we take the convention

$$R^\perp(X,Y)\eta = \nabla^\perp_X \nabla^\perp_Y \eta - \nabla^\perp_Y \nabla^\perp_X \eta - \nabla^\perp_{[X,Y]}\eta.$$

The Ricci equation reads

$$\langle R^\perp(X,Y)\eta, \zeta\rangle = g([A_\eta, A_\zeta]X, Y),$$

where η, ζ are normal vector fields and

$$[A_\eta, A_\zeta] = A_\eta A_\zeta - A_\zeta A_\eta.$$

Under the PNMC assumption, $\nabla^\perp \xi = 0$ implies

$$R^\perp(X,Y)\xi = 0.$$

Consequently,

$$[A_\xi, A_\eta] = 0 \qquad \text{for every normal vector field } \eta. \tag{2.3}$$

Thus every normal shape operator preserves the eigenspaces of A_ξ on U.

We shall also use the Gauss equation in the geometric reduction to express intrinsic curvature in terms of the second fundamental form. We use the convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

The Gauss equation is

$$g(R(X,Y)Z, W) = \langle B(X,W), B(Y,Z)\rangle - \langle B(X,Z), B(Y,W)\rangle, \tag{2.4}$$

for tangent vector fields X, Y, Z, W.

3. Proof of the theorem

Equation (2.1) singles out $\operatorname{grad} h$ as a principal direction wherever it is nonzero. In Subsection 3.1, we show that the integral curves in this direction are geodesics when parametrized by arclength and admit parallel principal frames. The Gauss and Codazzi equations along one such curve will then give the closed differential system in Proposition 1. Subsection 3.2 proves that this system has no solution with the required initial conditions.

3.1. **Geometric reduction.** Suppose, for a contradiction, that x is a nonminimal biharmonic immersion with PNMC. Then $U = \{H \neq 0\}$ is nonempty. The function h cannot be constant on a nonempty open subset of U, since (2.2) would force $A_\xi = 0$, contrary to $\operatorname{tr} A_\xi = mh > 0$. We may therefore restrict to a nonempty connected open set $\Omega \subset U$ on which $\operatorname{grad} h \neq 0$.

To make the principal curvature in the gradient direction positive, set

$$\nu = -\xi, \qquad L = \frac{m}{2}h, \qquad A = A_\nu = -A_\xi.$$

The trace identity and equations (2.1)–(2.2) become

$$\operatorname{tr} A = -2L, \qquad A \operatorname{grad} L = L \operatorname{grad} L, \qquad \Delta L = L|A|^2. \tag{3.1}$$

Here $L > 0$, $\operatorname{grad} L \neq 0$, and A is a symmetric Codazzi tensor. If $m = 1$, the middle identity gives $A = L \operatorname{Id}$, which contradicts $\operatorname{tr} A = -2L$. Hence we may assume $m \geq 2$.

To parametrize the integral curves of $\operatorname{grad} L$ by arclength, put

$$T = \frac{\operatorname{grad} L}{|\operatorname{grad} L|}.$$

Then

$$AT = LT, \qquad T(L) = |\operatorname{grad} L| > 0.$$

To express the geometric equations along these curves, we seek a principal frame that is parallel along T with respect to the induced connection ∇.

To choose smooth principal fields, restrict Ω to a neighbourhood of a point where A has the maximal number of distinct eigenvalues. Continuity keeps these eigenvalues separated, and maximality prevents further splitting, so their multiplicities are constant on a sufficiently small neighbourhood. The principal curvatures are then smooth, and their eigenspaces form smooth distributions. For the principal-frame assertions, we adapt the argument of Koiso and Urakawa [2018, Lemmas 4.3–4.4] to the Codazzi tensor A.

Lemma 1. *The eigenvalue L of A is simple. Parallel transport along the integral curves of T preserves each eigenspace of A. Moreover,*

$$\nabla_T T = 0, \qquad B(T, Y) = 0 \quad \textit{whenever } g(T, Y) = 0.$$

Proof. Suppose that L has multiplicity greater than one. Since its multiplicity is constant, we can choose a local unit L-eigenfield Y perpendicular to T. The Codazzi equation gives

$$T(L) = g((\nabla_T A)Y, Y) = g((\nabla_Y A)T, Y).$$

Differentiating $AT = LT$ in the direction Y, we obtain

$$(\nabla_Y A)T = Y(L)T + (L \operatorname{Id} - A)\nabla_Y T.$$

Its inner product with Y vanishes, since $Y \perp T$, $AY = LY$, and A is self-adjoint. The equality $T(L) = 0$ contradicts $T(L) > 0$, so L is simple.

To show that parallel transport along T preserves the eigenspaces, let Y, Z be local eigenfields corresponding to distinct principal curvatures ℓ, k. Differentiating $AY = \ell Y$ along T and taking the inner product with Z gives

$$(\ell - k) g(\nabla_T Y, Z) = g((\nabla_T A)Y, Z).$$

It therefore suffices to prove that $\nabla_T A$ commutes with A: this will force the right-hand side to vanish for every pair of distinct eigenspaces.

Differentiating $A \operatorname{grad} L = L \operatorname{grad} L$ in the direction of an arbitrary tangent field X gives

$$(\nabla_X A) \operatorname{grad} L + (A - L \operatorname{Id}) \nabla_X \operatorname{grad} L = X(L) \operatorname{grad} L.$$

The second term contains the Hessian of L, whose symmetry will be used below. Put $v = \operatorname{grad} L$, and let Q be the Hessian endomorphism of L, defined by $Q(X) = \nabla_X v$. By Codazzi,

$$(\nabla_X A)v = (\nabla_v A)X.$$

The differentiated relation becomes

$$\nabla_v A + (A - L \operatorname{Id})Q = v \otimes v,$$

where $(v \otimes v)X = g(v, X)v$.

The operators Q, $\nabla_v A$ and $v \otimes v$ are self-adjoint, so comparison with the adjoint identity gives

$$[A, Q] = 0.$$

The relation $Av = Lv$ also implies that A commutes with $v \otimes v$. Taking the commutator with A in the differentiated relation now gives

$$[A, \nabla_v A] = 0.$$

Since $v = |\operatorname{grad} L| T$, it follows that

$$[A, \nabla_T A] = 0.$$

The relation for $g(\nabla_T Y, Z)$ now shows that this inner product vanishes whenever Y and Z lie in distinct eigenspaces. Thus $\nabla_T Y$ remains in the eigenspace containing Y, and parallel transport along the integral curves of T preserves each principal distribution.

In particular, $\nabla_T T$ belongs to the simple L-eigenspace spanned by T. Since T has unit length, $\nabla_T T \perp T$, and therefore $\nabla_T T = 0$.

Finally, (2.3) and $A = -A_\xi$ give $[A, A_\eta] = 0$ for every normal field η. Thus $A_\eta T$ lies in the simple L-eigenspace, and for every tangent field $Y \perp T$,

$$\langle B(T, Y), \eta \rangle = g(A_\eta T, Y) = 0.$$

Since this holds for every normal field η, it follows that $B(T, Y) = 0$. □

We now derive a closed differential system along one of these geodesics. Fix an integral curve $\gamma : I \to \Omega$ of T, where I is an open interval containing 0 and s is the arclength parameter. Put $n = m - 1$. By Lemma 1, choose an orthonormal principal frame $T, E_1, \dots, E_n$ for A along γ such that

$$\nabla_T T = 0, \qquad \nabla_T E_i = 0.$$

After restricting I if necessary, we extend the fields E_i smoothly to a neighbourhood of γ as principal fields perpendicular to T. Write $AE_i = \ell_i E_i$, allowing repetitions among the principal curvatures ℓ_i. Primes denote differentiation with respect to s along γ.

The Gauss and Codazzi equations also involve $\nabla_{E_i} T$. Since $E_i \perp \operatorname{grad} L$, we have $E_i(L) = 0$, and Codazzi gives

$$\ell_i' E_i = (\nabla_T A)E_i = (\nabla_{E_i} A)T = (L \operatorname{Id} - A)\nabla_{E_i} T.$$

The vector $\nabla_{E_i}T$ is perpendicular to T. Since L is a simple eigenvalue, $L\,\mathrm{Id}-A$ is invertible on $T^\perp$. Hence

$$\nabla_{E_i}T=\mu_iE_i,\qquad \mu_i=\frac{\ell_i'}{L-\ell_i}.$$

To find μ_i', we use the Gauss equation for the planes spanned by E_i and T. Since $B(T,E_i)=0$, this equation involves only the normal vectors $B(T,T)$ and $B(E_i,E_i)$.

Choose a parallel orthonormal normal frame along γ, containing ν, to identify the normal spaces with $\mathbb{R}^r$. In these coordinates, ν is constant and normal covariant differentiation along γ is ordinary differentiation. Set

$$b(s)=B(T,T)\big|_{\gamma(s)},\qquad \lambda_i(s)=B(E_i,E_i)\big|_{\gamma(s)}.$$

Since $A=A_\nu$,

$$\langle b,\nu\rangle=L,\qquad \langle\lambda_i,\nu\rangle=\ell_i.$$

By Lemma 1, $\nabla_TT=0$ throughout the chosen neighbourhood. Along γ, we also have

$$[E_i,T]=\nabla_{E_i}T-\nabla_TE_i=\mu_iE_i.$$

Hence our curvature convention gives

$$\begin{aligned}R(E_i,T)T&=\nabla_{E_i}\nabla_TT-\nabla_T\nabla_{E_i}T-\nabla_{[E_i,T]}T\\&=-(\mu_i'+\mu_i^2)E_i.\end{aligned}$$

By the Gauss equation (2.4) and $B(T,E_i)=0$,

$$g(R(E_i,T)T,E_i)=\langle B(E_i,E_i),B(T,T)\rangle=\langle\lambda_i,b\rangle.$$

Therefore

$$\mu_i'=-\mu_i^2-\langle b,\lambda_i\rangle.$$

For the vectors λ_i in this equation, Codazzi and $\nabla_TE_i=0$ give

$$\begin{aligned}\lambda_i'&=(\nabla^\perp_TB)(E_i,E_i)\\&=(\nabla^\perp_{E_i}B)(T,E_i)\\&=-B(\nabla_{E_i}T,E_i)-B(T,\nabla_{E_i}E_i).\end{aligned}$$

Here $B(T,E_i)$ vanishes identically by Lemma 1. Also,

$$g(\nabla_{E_i}E_i,T)=-g(E_i,\nabla_{E_i}T)=-\mu_i.$$

Using $B(T,Y)=0$ for $Y\perp T$, we obtain

$$\lambda_i'=\mu_i(b-\lambda_i).$$

The equations for λ_i and μ_i still contain b. To express this vector in terms of the chosen unknowns, take the trace of B:

$$b+\sum_{i=1}^n\lambda_i=mH=-2L\nu.$$

Taking the trace of A also gives

$$L+\sum_{i=1}^n\ell_i=-2L.$$

Consequently,

$$L=-\frac{1}{3}\sum_{i=1}^n\ell_i,\qquad b=-\sum_{i=1}^n\lambda_i-2L\nu.$$

These expressions close the first-order system for (λ_i,μ_i). The other components $B(E_i,E_j)$, $i\neq j$, need not vanish; they do not enter these equations.

It remains to use the scalar equation $\Delta L = L|A|^2$ from (3.1). On Ω, the definition of T gives

$$\operatorname{grad} L = T(L)T,$$

and therefore

$$\Delta L = T(T(L)) + T(L)\operatorname{div} T.$$

Along γ, we have $T(L) = L'$, $T(T(L)) = L''$, and

$$\operatorname{div} T = g(\nabla_T T, T) + \sum_{i=1}^{n} g(\nabla_{E_i} T, E_i) = \sum_{i=1}^{n} \mu_i.$$

Consequently,

$$\Delta L = L'' + \left(\sum_{i=1}^{n} \mu_i\right) L'.$$

Since the frame is principal,

$$|A|^2 = L^2 + \sum_i \ell_i^2,$$

and the scalar equation becomes

$$L'' + \left(\sum_{i=1}^{n} \mu_i\right) L' = L\left(L^2 + \sum_{i=1}^{n} \ell_i^2\right).$$

Moreover, $L > 0$ and $L' = |\operatorname{grad} L| > 0$ along γ.

These calculations give the following necessary conditions.

Proposition 1. *If $x : M^m \to \mathbb{E}^{m+r}$ is a nonminimal biharmonic immersion with PNMC, then $m \geq 2$. With $n = m - 1$, there exist an open interval I containing 0, a fixed unit vector $\nu \in \mathbb{R}^r$, and smooth functions*

$$\lambda_i : I \to \mathbb{R}^r, \qquad \mu_i : I \to \mathbb{R}, \qquad 1 \leq i \leq n,$$

with the following properties. Define

$$\ell_i = \langle \lambda_i, \nu \rangle, \qquad L = -\frac{1}{3}\sum_{i=1}^{n} \ell_i, \qquad b = -\sum_{i=1}^{n} \lambda_i - 2L\nu. \tag{3.2}$$

Then

$$\lambda_i' = (b - \lambda_i)\mu_i, \qquad 1 \leq i \leq n, \tag{3.3a}$$

$$\mu_i' = -\mu_i^2 - \langle b, \lambda_i \rangle, \qquad 1 \leq i \leq n, \tag{3.3b}$$

$$L'' + \left(\sum_{i=1}^{n} \mu_i\right) L' = L\left(L^2 + \sum_{i=1}^{n} \ell_i^2\right), \tag{3.3c}$$

and $L(0) > 0$, $L'(0) > 0$.

In codimension one, this system agrees, up to conventions of sign and normalization, with that obtained by Koiso and Urakawa [2018, Proposition 4.7]. Proposition 1 gives the corresponding reduction for PNMC submanifolds in arbitrary codimension.

After substituting (3.2), equations (3.3a)–(3.3b) form a polynomial autonomous system for (λ_i, μ_i). Equation (3.3c) is an additional condition on its solutions.

3.2. **Nonexistence of solutions.** To exclude the solutions in Proposition 1, we consider λ_i, μ_i as functions of s, independently of the immersion, retaining (3.2).

Proposition 2. *For any $n, r \geq 1$ and any fixed unit vector $\nu \in \mathbb{R}^r$, the system* (3.3) *admits no solution on an open interval containing* 0 *with $L(0) > 0$ and $L'(0) > 0$.*

Proof. Suppose, for a contradiction, that such a solution exists. We shall obtain a uniform positive lower bound for L' and an upper bound for L on an infinite forward interval. To exclude a finite endpoint of that interval, we shall need bounds for every component of the first-order system.

The given solution of the polynomial first-order system (3.3a)–(3.3b) extends uniquely to a maximal interval J containing 0 [Teschl, 2012, Section 2.6]. Let $s_* \in (0, \infty]$ be its forward endpoint.

We must check that this extension also satisfies the scalar equation (3.3c). Solutions of the polynomial system are real analytic [Teschl, 2012, Section 4.1]. Since L and the ℓ_i are linear combinations of components of the λ_i, the expression

$$L'' + \left(\sum_{i=1}^{n} \mu_i\right) L' - L\left(L^2 + \sum_{i=1}^{n} \ell_i^2\right)$$

is real analytic on J. It vanishes on the original open interval and hence throughout J. This continuation concerns the ODE solution and does not require an extension of the original immersion.

We work on $0 \leq s < s_*$. To control the components of the first-order system, consider the squared norm of each pair (λ_i, μ_i). The coupling terms involving b cancel upon differentiation:

$$\begin{aligned}\left(\mu_i^2 + |\lambda_i|^2\right)' &= 2\mu_i\left(-\mu_i^2 - \langle b, \lambda_i\rangle\right) + 2\mu_i\langle \lambda_i, b - \lambda_i\rangle \\ &= -2\mu_i\left(\mu_i^2 + |\lambda_i|^2\right).\end{aligned}$$

This homogeneous linear equation shows that each pair is either identically zero or has positive squared norm throughout J. Identically zero pairs contribute nothing to the equations or trace identities, so we omit them and relabel the rest. Restoring these pairs gives a solution of the original first-order system, so the maximal interval is unchanged. From now on, n counts the remaining pairs; $n \geq 1$ by the trace identity and $L(0) > 0$.

The coefficient of L' in the scalar biharmonic equation is $\sum_i \mu_i$. To obtain an integrating factor, we seek positive functions with logarithmic derivatives μ_i. Taking inverse square roots in the preceding norm identity gives precisely such functions:

$$D_i = \frac{1}{\sqrt{\mu_i^2 + |\lambda_i|^2}} > 0, \qquad D_i' = \mu_i D_i.$$

Thus the positive function

$$W = \prod_{i=1}^{n} D_i$$

satisfies $W'/W = \sum_i \mu_i$, and the scalar equation becomes

$$(WL')' = WL\left(L^2 + \sum_{i=1}^{n} \ell_i^2\right).$$

As long as $L > 0$, its right-hand side is positive, and hence

$$WL' \geq W(0)L'(0) =: c_0 > 0.$$

Since $W > 0$, we have $L' > 0$ as long as $L > 0$; hence L cannot reach zero. Thus, for every $0 \le s < s_*$,

$$L(s) \ge L(0) > 0, \qquad L'(s) > 0, \qquad W(s)L'(s) \ge c_0. \tag{3.4}$$

To obtain a uniform positive lower bound for L' from (3.4), it suffices to bound the factors D_i of W above. By definition,

$$|D_i'| = \frac{|\mu_i|}{\sqrt{\mu_i^2 + |\lambda_i|^2}} \le 1, \qquad |\ell_i D_i| \le 1,$$

where the second inequality follows from $|\ell_i| \le |\lambda_i|$. We shall relate D_i to L through the components ℓ_i. From (3.2),

$$\langle b, \nu \rangle = -\sum_{i=1}^{n} \ell_i - 2L = L.$$

Projecting the equation for λ_i onto ν therefore gives

$$\ell_i' = (L - \ell_i)\mu_i.$$

Together with $D_i' = \mu_i D_i$, this gives

$$(L - \ell_i)' = L' - (L - \ell_i)\mu_i, \qquad D_i' = \mu_i D_i.$$

The terms involving μ_i cancel in the derivative of the product. With

$$\Phi_i = (L - \ell_i)D_i,$$

we obtain

$$\Phi_i' = L' D_i > 0, \tag{3.5}$$

so every Φ_i is strictly increasing.

Since $|\ell_i D_i| \le 1$,

$$LD_i = \Phi_i + \ell_i D_i \le \Phi_i + 1, \qquad D_i \le \frac{\Phi_i + 1}{L}.$$

Although Φ_i is increasing, the quotient on the right is nonincreasing:

$$\left(\frac{\Phi_i + 1}{L}\right)' = \frac{L'}{L^2}(LD_i - \Phi_i - 1) = \frac{L'}{L^2}(\ell_i D_i - 1) \le 0.$$

Consequently,

$$0 < D_i(s) \le C_i, \qquad C_i := \frac{\Phi_i(0) + 1}{L(0)} > 0. \tag{3.6}$$

Since $W \le \prod_i C_i$, (3.4) gives the uniform lower bound

$$L'(s) \ge \frac{c_0}{\prod_{i=1}^{n} C_i} =: c_1 > 0. \tag{3.7}$$

To exclude a finite forward endpoint, we need bounds for the full solution. We first bound all but one pair (λ_i, μ_i), then use the trace relation to bound L, which will in turn control the remaining pair. Since

$$\mu_i^2 + |\lambda_i|^2 = D_i^{-2},$$

we need positive lower bounds for the D_i. A bound on the decrease of $\log D_i$ will give such a lower bound, so we first compare the logarithmic derivatives.

For distinct indices i, j, the bounds (3.4) and (3.6) give

$$\begin{aligned} |(\log D_j)'| = |\mu_j| \le \frac{1}{D_j} \le \frac{L'}{c_0} \prod_{k \neq j} D_k \\ \le C_{ij} L' D_i = C_{ij} \Phi_i', \end{aligned} \tag{3.8}$$

where

$$C_{ij} = c_0^{-1} \prod_{k \neq i,j} C_k > 0.$$

Thus bounded increase of a single Φ_i gives positive lower bounds for all D_j with $j \neq i$. The trace relation and monotonicity supply such an index.

At each s, the identity $\sum_i \ell_i = -3L$ implies that some index i satisfies

$$\ell_i \le -\frac{3L}{n} < 0.$$

For that index, $D_i \le 1/|\ell_i|$, and hence

$$0 < \Phi_i = (L - \ell_i) D_i \le \frac{L - \ell_i}{|\ell_i|} = 1 + \frac{L}{|\ell_i|} \le 1 + \frac{n}{3}.$$

Put $K = 1 + n/3$.

The index may depend on s. Choose an increasing sequence (s_k) such that $s_k \to s_*$. Since there are only finitely many indices, one index satisfies the bound along an infinite subsequence; relabel it as 1. The function Φ_1 is increasing and is bounded above by K along a sequence tending to s_*, so it is bounded above by K throughout the forward interval. Choose one point s_0 of that subsequence and put $a = \Phi_1(s_0) > 0$. Then

$$a \le \Phi_1(s) \le K \qquad (s_0 \le s < s_*). \tag{3.9}$$

For $j \neq 1$, integration of (3.8) with $i = 1$ from s_0 to s gives

$$\left| \log \frac{D_j(s)}{D_j(s_0)} \right| \le C_{1j} (\Phi_1(s) - \Phi_1(s_0)) \le C_{1j}(K - a).$$

In particular,

$$D_j(s) \ge D_j(s_0) e^{-C_{1j}(K-a)} > 0 \qquad (s_0 \le s < s_*).$$

Thus λ_j, μ_j and ℓ_j are bounded for every $j \neq 1$.

The remaining pair requires a lower bound for D_1. Since

$$D_1 = \frac{\Phi_1}{L - \ell_1}, \qquad \Phi_1 \ge a > 0,$$

it is enough to bound $L - \ell_1$ from above. The trace relation gives

$$L - \ell_1 = 4L + \sum_{j \neq 1} \ell_j.$$

The sum is already bounded, so it remains to obtain an upper bound for L.

Choose a constant $C_\ell \ge 0$ such that

$$\left| \sum_{j \neq 1} \ell_j \right| \le C_\ell.$$

If $n = 1$, this sum is zero and we take $C_\ell = 0$. Since $\Phi_1 = (L - \ell_1) D_1 \ge a > 0$, we have

$$0 < L - \ell_1 \le 4L + C_\ell.$$

Identity (3.5) therefore allows us to compare the growth of L with that of Φ_1:

$$\frac{L'}{4L + C_\ell} \le \frac{L'}{L - \ell_1} = \frac{\Phi_1'}{\Phi_1}.$$

Integrating yields

$$\frac{1}{4} \log \frac{4L(s) + C_\ell}{4L(s_0) + C_\ell} \le \log \frac{\Phi_1(s)}{\Phi_1(s_0)} \le \log \frac{K}{a}.$$

It follows that L is bounded on $[s_0, s_*)$.

The bound for L now gives

$$D_1 = \frac{\Phi_1}{L - \ell_1} \ge \frac{a}{4L + C_\ell},$$

so D_1 is bounded away from zero and the remaining pair (λ_1, μ_1) is bounded. Together with the bounds for $j \neq 1$, this bounds every component of the first-order system on $[s_0, s_*)$.

If $s_* < \infty$, the polynomial right-hand sides are bounded along the solution, so its derivatives are bounded and it has a finite limit as $s \to s_*$ from below. Local existence and uniqueness extend the solution through this limit, contrary to the maximality of J. Thus $s_* = \infty$.

The function L is bounded on $[s_0, \infty)$. On the other hand, the lower bound (3.7) gives

$$L(s) \ge L(s_0) + c_1(s - s_0),$$

which tends to infinity as $s \to \infty$. This contradiction proves the proposition. □

Proof of the theorem. Let $x : M^m \to \mathbb{E}^{m+r}$ be a biharmonic immersion with PNMC. If it were not minimal, Proposition 1 would produce a solution of the differential system with $L(0) > 0$ and $L'(0) > 0$, contradicting Proposition 2. Hence H vanishes identically, and the immersion is minimal. □

Proof of the corollary. For a hypersurface, $H/|H|$ is a unit normal wherever $H \neq 0$, and is therefore parallel in the normal bundle. Every biharmonic Euclidean hypersurface thus satisfies the PNMC condition and is minimal by the theorem. □

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